

RESIDUE CLASSES CONTAINING AN UNEXPECTED NUMBER OF PRIMES

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ABSTRACT. We fix a non-zero integer a and consider arithmetic progressions $a \bmod q$, with q varying over a given range. We show that for certain specific values of a , the arithmetic progressions $a \bmod q$ contain, on average, significantly fewer primes than expected.

1. INTRODUCTION

The prime number theorem for arithmetic progressions asserts that

$$\psi(x; q, a) \sim \psi(x)/\phi(q)$$

for any a and q such that $(a, q) = 1$. Another way to say this is that the primes are equidistributed in the $\phi(q)$ arithmetic progressions $a \bmod q$ with $(a, q) = 1$.

Fix an integer $a \neq 0$. We will be interested in the number of primes in the arithmetic progressions $a \bmod q$ with q varying in certain ranges, and we will show that for specific values of a , there are significantly fewer primes in these arithmetic progressions than in typical arithmetic progressions. Consider the average value of $\psi(x; q, a) - \psi(x)/\phi(q)$ over q . One might expect that no matter what the value of a is, the cancellations in these oscillating terms will force the average to be very small. However it turns out that the average is highly dependent on the arithmetical properties of a .

Here is the main result of the paper.

Theorem 1.1. *Fix an integer $a \neq 0$ and let $M = M(x) \leq (\log x)^B$ where $B > 0$ is a fixed real number. The average error term in the usual approximation for the number of primes $p \equiv a \bmod q$ with $p \leq x$, $p \neq a$, where $(q, a) = 1$ and $q \leq x/M$, is*

$$\frac{1}{\frac{\phi(a)}{a} \frac{x}{M}} \sum_{\substack{q \leq \frac{x}{M} \\ (q, a) = 1}} \left(\psi(x; q, a) - \Lambda(a) - \frac{\psi(x)}{\phi(q)} \right) = \mu(a, M) + O_{a, \epsilon, B} \left(\frac{1}{M^{\frac{205}{538} - \epsilon}} \right) \quad (1)$$

with

$$\mu(a, M) := \begin{cases} 0 & \text{if } \omega(a) \geq 2 \\ -\frac{1}{2} \log p & \text{if } a = \pm p^e \\ -\frac{1}{2} \log M - C_5 & \text{if } a = \pm 1, \end{cases}$$

where

$$C_5 := \frac{1}{2} \left(\log 2\pi + \gamma + \sum_p \frac{\log p}{p(p-1)} + 1 \right).$$

Remark 1.2. Assuming Lindelöf's hypothesis, we can replace the error term in (1) by $O_{a, \epsilon, B} \left(\frac{1}{M^{1/2 - \epsilon}} \right)$.

Remark 1.3. We subtracted $\Lambda(a)$ from $\psi(x; q, a)$ in (1) because the arithmetic progression $a \bmod q$ contains the prime power p^e for all q if $a = p^e$.

Remark 1.4. It may be preferable to replace $\psi(x)$ by $\psi(x, \chi_0)$ in Theorem 1.1, since the quantity

$$\psi(x; q, a) - \psi(x, \chi_0)/\phi(q)$$

is the discrepancy (with signs) of the sequence of primes in the reduced residue classes mod q . One can do this with a negligible error term.

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2. PAST RESULTS

The study of the discrepancy $\psi(x; q, a) - x/\phi(q)$ on average has been a fruitful subject over the past decades. For example, the celebrated theorem of Bombieri-Vinogradov gives a bound on the sum of the mean absolute value of the maximum of this discrepancy over all $1 \leq a < q$ with $(a, q) = 1$, summed over $q \leq x^{1/2-o(1)}$. The Hooley-Montgomery refinement of the Barban-Davenport-Halberstam Theorem gives an estimation of the variance of $\psi(x; q, a) - x/\phi(q)$, again for all values of a in the range $1 \leq a < q$ with $(a, q) = 1$ for $q < x/(\log x)^A$. The mean value of $\psi(x; q, a) - x/\phi(q)$ was studied for fixed values of a for $q \geq x^{1/2}$ (see [1],[3] or [4]), and bounds on this mean value turned out to be applicable to Titchmarsh's divisor problem, first solved by Linnik. The best result so far for this problem was obtained by Friedlander and Granville.

Theorem 2.1 (Friedlander, Granville). *Let $0 < \lambda < 1/4$, $A > 0$ be given. Then uniformly for $0 < |a| < x^\lambda$, $2 \leq Q \leq x/3$ we have*

$$\sum_{\substack{Q < q \leq 2Q \\ (q, a) = 1}} \left(\psi(x; q, a) - \frac{x}{\phi(q)} \right) \ll_{\lambda, A} 2^{\omega(a)} Q \log(x/Q) + \frac{x}{(\log x)^A} + Q \log |a|. \quad (2)$$

Remark 2.2. If a is not a prime power the term $Q \log |a|$ may be deleted.

Theorem 2.1 is a refinement of the deep results of Bombieri-Friedlander-Iwaniec [1] and of Fouvry [3], and makes use of the dispersion method combined with Fourier analysis and involved estimates on Kloosterman sums.

The main method used in our paper, which we will refer to as the "divisor switching" technique, stemmed from the work of Dirichlet on the divisor problem. Variants of his "hyperbola method" were subsequently used in many different contexts, and have become a very important tool in analytic number theory. The variant which will be used in this paper is very similar to that of Hooley [6].

3. MAIN RESULTS

What we will actually prove in Section 6 is a uniform version of Theorem 1.1 (which is too technical to state at this point), and we will see that the proportion of integers $|a| \leq x^{\frac{1}{4}-\epsilon}$ for which we get an error term of $O(M^{-\eta})$ in (1) is at least $(1-4\eta)e^{-\gamma}$ (here and throughout, γ denotes the Euler-Mascheroni constant).

A number of constants will appear in the paper.

Definition 3.1. *We define*

$$C_1(a) := \frac{\zeta(2)\zeta(3)}{\zeta(6)} \frac{\phi(a)}{a} \prod_{p|a} \left(1 - \frac{1}{p^2 - p + 1}\right),$$

$$C_3(a) := C_1(a) \left(\gamma - 1 - \sum_p \frac{\log p}{p^2 - p + 1} + \sum_{p|a} \frac{p^2 \log p}{(p-1)(p^2 - p + 1)} \right),$$

$$C_5 := \frac{1}{2} \left(\log 2\pi + \gamma + \sum_p \frac{\log p}{p(p-1)} + 1 \right).$$

We will denote by $\omega(a)$ the number of distinct prime factors of a . Note that provided $0 \neq |a| \leq x$ and $\sum_{d|a} 1 = o\left(\frac{x}{M} \frac{\phi(a)}{a}\right)$, there are $\sim \frac{\phi(a)}{a} \frac{x}{M}$ terms in the sum over $1 \leq q \leq x/M$ with the condition $(q, a) = 1$.

The results of Theorem 1.1 become worse as M gets smaller. The reason for this is that when $M = O(1)$ (that is the relevant arithmetic progressions have bounded length), the estimates become dramatically different. This case is handled in the next Proposition.

Proposition 3.2. *Fix A and $\lambda < \frac{1}{4}$, two positive real numbers. For a in the range $0 < |a| \leq x^\lambda$ such that $\omega(a) \leq 10 \log \log x$ we have*

$$\frac{1}{\frac{\phi(a)}{a} x} \sum_{\substack{q \leq x \\ (q, a) = 1}} \left(\psi(x; q, a) - \Lambda(a) - \frac{\psi(x)}{\phi(q)} \right) = \frac{a}{\phi(a)} C_3(a) + O_{A, \lambda} \left(\frac{1}{(\log x)^A} \right). \quad (3)$$

More generally, for $M \geq 1$ a fixed integer we have

$$\frac{1}{\frac{\phi(a)}{a} M} \sum_{\substack{q \leq \frac{x}{M} \\ (q, a) = 1}} \left(\psi(x; q, a) - \Lambda(a) - \frac{\psi(x)}{\phi(q)} \right) = \mu'(a, M) + O_{A, \lambda} \left(\frac{1}{(\log x)^A} \right), \quad (4)$$

where

$$\mu'(a, M) := \frac{a}{\phi(a)} M \left(C_1(a) \log M + C_3(a) - \sum_{\substack{r \leq M \\ (r, a) = 1}} \frac{1}{\phi(r)} \left(1 - \frac{r}{M}\right) \right).$$

Note that $\mu'(a, 1) = \frac{a}{\phi(a)} C_3(a)$.

Remark 3.3. It is possible to give such an estimate for any $M \in \mathbb{R}_{\geq 1}$, however the formula is simpler when M is an integer, so we preferred to just include the latter case.

By inverting the order of summation in Proposition 3.2, one gets the following corollary, which is an example of application of the results of Bombieri, Fouvry, Friedlander, Granville and Iwaniec (see [1], [3] and [4]).

Corollary 3.4 (Titchmarsh's divisor problem). *Fix A and $\lambda < \frac{1}{4}$, two positive real numbers. For a in the range $0 < |a| \leq x^\lambda$ such that $\omega(a) \leq 10 \log \log x$ we have*

$$\sum_{|a| < n \leq x} \Lambda(n) \tau(n-a) = C_1(a)x \log x + (2C_3(a) + C_1(a))x + O_{A,\lambda} \left(\frac{x}{(\log x)^A} \right). \quad (5)$$

Note that the constant 10 in Proposition 3.2 and Corollary 3.4 can be replaced by an arbitrarily large real number.

4. NOTATION

Definition 4.1. *For $n \neq 0$ an integer (possibly negative), we define*

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^e \\ 0 & \text{otherwise,} \end{cases} \quad \vartheta(n) := \begin{cases} \log p & \text{if } n = p \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.2.

$$\psi(x; q, a) := \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n), \quad \theta(x; q, a) := \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \vartheta(n).$$

The following definition is non-standard but will be useful in the proofs.

Definition 4.3.

$$\psi^*(x; q, a) := \sum_{\substack{|a| < n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n), \quad (6)$$

$$\theta^*(x; q, a) := \sum_{\substack{|a| < n \leq x \\ n \equiv a \pmod{q}}} \vartheta(n). \quad (7)$$

We will need to consider the prime divisors of a which are less than M .

Definition 4.4. *For a an integer and $M > 0$ a real number, we define*

$$a_M := \prod_{\substack{p|a \\ p \leq M}} p. \quad (8)$$

The error term $E(M, a)$ will be defined depending on the context, so one has to pay attention to its definition in every statement.

5. LEMMAS

We begin by recalling the divisor switching technique.

Lemma 5.1. *Let a be an integer such that $0 < |a| \leq x$ and let $M = M(x)$ such that $1 \leq M < x$. We have*

$$\sum_{\substack{\frac{x}{M} < q \leq x \\ (q,a)=1}} \sum_{\substack{|a| < p \leq x \\ p \equiv a \pmod{q}}} \log p = \sum_{\substack{1 \leq r < (x-a) \frac{M}{x} \\ (r,a)=1}} \sum_{\substack{r \frac{x}{M} + a < p \leq x \\ p \equiv a \pmod{r} \\ p > |a|}} \log p + O(|a| \log x). \quad (9)$$

Proof. Clearly,

$$\sum_{\substack{\frac{x}{M} < q \leq x \\ (q,a)=1}} \sum_{\substack{|a| < p \leq x \\ p \equiv a \pmod{q}}} \log p = \sum_{\substack{\frac{x}{M} < q \leq x \\ (q,a)=1}} \sum_{\substack{|a| < p \leq x \\ p \equiv a \pmod{q} \\ p > a + \frac{x}{M}}} \log p. \quad (10)$$

We now apply Hooley's variant of the divisor switching technique (see [6]). Setting $p - a = rq$ in (10), one can sum over r instead of summing over q . Now $(r, a) = 1$, else $(p, a) > 1$ so $p \mid a$, but this is impossible since $p > |a|$. Taking $a > 0$ for now, we get that (10) is equal to

$$\sum_{\substack{\frac{x}{M} < q \leq x-a \\ (q,a)=1}} \sum_{\substack{|a| < p \leq x \\ p \equiv a \pmod{q} \\ p > a + \frac{x}{M}}} \log p = \sum_{\substack{1 \leq r < (x-a) \frac{M}{x} \\ (r,a)=1}} \sum_{\substack{r \frac{x}{M} + a < p \leq x \\ p \equiv a \pmod{r} \\ p > |a|}} \log p. \quad (11)$$

If we had $a < 0$, additional terms would be needed in passing from the right hand side of (10) to the left hand side of (11). These additional terms are

$$\ll \sum_{x < q \leq x-a} \log x \ll |a| \log x.$$

The proof is complete. □

Lemma 5.2. *We have the following estimates:*

$$\sum_{\substack{n \leq M \\ (n,a)=1}} \frac{1}{\phi(n)} = C_1(a) \log M + C_1(a) + C_3(a) + O\left(2^{\omega(a)} \frac{\log M}{M}\right), \quad (12)$$

$$\sum_{\substack{n \leq M \\ (n,a)=1}} \frac{n}{\phi(n)} = C_1(a)M + O\left(2^{\omega(a)} \log M\right). \quad (13)$$

Note that without loss of generality, we can replace a by a_M on the right side of (12) and (13).

Proof. The proof of (12) is very similar that of Lemma 13.1 in [5]. One first has to prove the following estimate:

$$\sum_{\substack{n \leq M \\ (n,a)=1}} \frac{1}{n} = \frac{\phi(a)}{a} \left(\log M + \gamma + \sum_{p|a} \frac{\log p}{p-1} \right) + O\left(\frac{2^{\omega(a)}}{M}\right). \quad (14)$$

One then writes

$$\sum_{\substack{n \leq M \\ (n,a)=1}} \frac{1}{\phi(n)} = \sum_{\substack{n \leq M \\ (n,a)=1}} \frac{1}{n} \sum_{d|n} \frac{\mu^2(d)}{\phi(d)} = \sum_{\substack{d \leq M \\ (d,a)=1}} \frac{\mu^2(d)}{d\phi(d)} \sum_{\substack{r \leq M/d \\ (r,a)=1}} \frac{1}{r} \quad (15)$$

and inserts the estimate (14) into (15). The final step is to bound the tail of the sums and to compute the following constants:

$$\sum_{(d,a)=1} \frac{\mu^2(d)}{d\phi(d)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_{p|a} \left(1 - \frac{1}{p^2 - p + 1}\right),$$

$$\sum_{(d,a)=1} \frac{\mu^2(d)}{d\phi(d)} \log d = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_{p|a} \left(1 - \frac{1}{p^2 - p + 1}\right) \sum_{p \nmid a} \frac{\log p}{p^2 - p + 1}.$$

The proof of (13) goes along the same lines. \square

The delicacy of the analysis forces us to give some details about the "trivial" estimates for the prime counting functions.

Lemma 5.3. *For any real number $\epsilon > 0$ and integer a with $0 < |a| \leq x$,*

$$\sum_{\substack{q \leq x \\ (q,a)=1}} (\psi^*(x; q, a) - \theta^*(x; q, a)) \ll_{\epsilon} x^{1/2+\epsilon}. \quad (16)$$

Proof.

$$\begin{aligned} & \sum_{\substack{q \leq x \\ (q,a)=1}} (\psi^*(x; q, a) - \theta^*(x; q, a)) \\ & \leq \sum_{q \leq x} \sum_{\substack{|a| < p^e \leq x \\ p^e \equiv a \pmod{q} \\ e \geq 2}} \log p \leq \sum_{2 \leq e \leq \frac{\log x}{\log 2}} \sum_{p \leq x^{1/e}} \sum_{\substack{q \leq x \\ q|p^e - a}} \log p \\ & \leq \log x \sum_{2 \leq e \leq \frac{\log x}{\log 2}} \sum_{p \leq x^{1/e}} \tau(p^e - a) \ll_{\epsilon} x^{\epsilon/2} \sum_{2 \leq e \leq \frac{\log x}{\log 2}} \pi(x^{1/e}) \\ & \ll_{\epsilon} x^{1/2+\epsilon}. \end{aligned}$$

\square

Lemma 5.4. *Let a be an integer with $0 \neq |a| \leq x$ and let $1 \leq Q \leq x$. We have*

$$\sum_{\substack{q \leq Q \\ (q,a)=1}} (\psi(x; q, a) - \Lambda(a) - \psi^*(x; q, a)) = O(|a|(\log |a|)^2), \quad (17)$$

$$\sum_{\substack{q \leq Q \\ (q,a)=1}} (\theta(x; q, a) - \vartheta(a) - \theta^*(x; q, a)) = O(|a|(\log |a|)^2). \quad (18)$$

Proof. Note that as soon as $q > 2|a|$, there are no integers congruent to $a \bmod q$ in the interval $[1, |a|)$. We then have

$$\begin{aligned} \sum_{\substack{q \leq Q \\ (q,a)=1}} (\psi(x; q, a) - \Lambda(a) - \psi^*(x; q, a)) &= \sum_{\substack{q \leq Q \\ (q,a)=1}} \sum_{\substack{1 \leq n < |a| \\ n \equiv a \bmod q}} \Lambda(n) \\ &\leq \sum_{\substack{q \leq 2|a| \\ (q,a)=1}} \log |a| \sum_{\substack{1 \leq n < |a| \\ n \equiv a \bmod q}} 1 \ll \sum_{\substack{q \leq 2|a| \\ (q,a)=1}} \log |a| \left(\frac{|a|}{q} \right) \ll |a| (\log |a|)^2. \end{aligned}$$

The proof for θ and θ^* is similar. □

Lemma 5.5. *Let $I \subset [1, x] \cap \mathbb{N}$. We have*

$$\sum_{q \in I} \left(\frac{x}{\phi(q)} - \frac{\psi(x)}{\phi(q)} \right) \ll x e^{-C\sqrt{\log x}}, \quad (19)$$

where C is an absolute positive constant.

Proof. This follows from Lemma 5.2 and the prime number theorem. □

To prove Lemma 5.9 we will need bounds on $\zeta(s)$.

Lemma 5.6. *Define $\theta := \frac{32}{205}$ and take any $\epsilon > 0$. In the region $|\sigma + it - 1| > \frac{1}{10}$, we have*

$$\zeta(\sigma + it) \ll_{\epsilon} (|t| + 1)^{\mu(\sigma) + \epsilon},$$

where

$$\mu(\sigma) = \begin{cases} 1/2 - \sigma & \text{if } \sigma \leq 0 \\ 1/2 + (2\theta - 1)\sigma & \text{if } 0 \leq \sigma \leq 1/2 \\ 2\theta(1 - \sigma) & \text{if } 1/2 \leq \sigma \leq 1 \\ 0 & \text{if } \sigma \geq 1. \end{cases}$$

Proof. For the values outside the critical strip, see for example section II.3.4 of [8]. In the critical strip, we use an estimate due to Huxley [7], which showed that $\zeta(1/2 + it) \ll_{\epsilon} (|t| + 1)^{\frac{32}{205} + \epsilon}$. The lemma then follows by convexity of μ . □

Remark 5.7. Under Lindelöf's hypothesis, the conclusion of Lemma 5.6 holds with $\theta = 0$.

Lemma 5.8 (Perron's formula). *Let $0 < \kappa < 1$, $y > 0$ and define*

$$h(y) := \begin{cases} 0 & \text{if } 0 < y < 1 \\ 1 - \frac{1}{y} & \text{if } y \geq 1. \end{cases}$$

We have

$$h(y) = \frac{1}{2\pi i} \int_{(\kappa)} \frac{y^s}{s(s+1)} ds.$$

Moreover, for $T \geq 1$ and positive $y \neq 1$, we have the estimate

$$h(y) = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} \frac{y^s}{s(s+1)} ds + O\left(\frac{y^{\kappa}}{T^2 |\log y|}\right).$$

Finally, for $y = 1$,

$$0 = h(1) = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \frac{ds}{s(s+1)} + O\left(\frac{1}{T}\right).$$

Proof. The first assertion is an easy application of the residue theorem.

Now take $y > 1$. We have again by the residue theorem that for any large integer $K \geq 3$ and for $T \geq 1$,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \frac{y^s}{s(s+1)} ds - h(y) &= \frac{1}{2\pi i} \left(\int_{\kappa-iT}^{\kappa-K-iT} + \int_{\kappa-K-iT}^{\kappa-K+iT} + \int_{\kappa-K+iT}^{\kappa+iT} \right) \frac{y^s}{s(s+1)} ds \\ &\ll \frac{1}{T^2} \int_{\kappa-K}^{\kappa} y^\sigma d\sigma + \frac{y^{\kappa-K}}{|\kappa-K|^2} \int_{\kappa-K-iT}^{\kappa-K+iT} |ds| \ll \frac{y^\kappa}{T^2 |\log y|} + T \frac{y^{\kappa-K}}{|\kappa-K|^2}. \end{aligned}$$

We deduce the second assertion of the lemma by letting K tend to infinity. The proof is similar in the case $0 < y < 1$.

The last case remaining is for $y = 1$. We have

$$\begin{aligned} \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \frac{ds}{s(s+1)} &= \frac{1}{2\pi i} \log \left(\frac{1 + \frac{1}{\kappa-iT}}{1 + \frac{1}{\kappa+iT}} \right) \\ &= \frac{1}{2\pi i} \log \left(1 + O\left(\frac{1}{T}\right) \right) \\ &= O\left(\frac{1}{T}\right), \end{aligned}$$

which concludes the proof. □

The following is a crucial lemma estimating a weighted sum of the reciprocal of the totient function.

Lemma 5.9. *Let $a \neq 0$ be an integer and $M \geq 1$ be a real number.*

If $\omega(a_M) \geq 1$,

$$\sum_{\substack{n \leq M \\ (n,a)=1}} \frac{1}{\phi(n)} \left(1 - \frac{n}{M}\right) = C_1(a_M) \log M + C_3(a_M) + \frac{\phi(a_M)}{a_M} \frac{\Lambda(a_M)}{2M} + E(M, a). \quad (20)$$

If $a_M = 1$,

$$\sum_{\substack{n \leq M \\ (n,a)=1}} \frac{1}{\phi(n)} \left(1 - \frac{n}{M}\right) = C_1(1) \log M + C_3(1) + \frac{1}{2} \frac{\log M}{M} + \frac{C_5}{M} + E(M, a). \quad (21)$$

There exists $\delta > 0$ such that the error term $E(M, a)$ satisfies

$$E(M, a) \ll_\epsilon \frac{\prod_{p|a_M} \left(1 + \frac{1}{p^\delta}\right)}{M} \left(\frac{a_M}{M}\right)^{\frac{205}{538} - \epsilon}. \quad (22)$$

Remark 5.10. Under Lindelöf's hypothesis,

$$E(M, a) \ll_{\epsilon} \frac{\prod_{p|a_M} \left(1 + \frac{1}{p^s}\right)}{M} \left(\frac{a_M}{M}\right)^{1/2-\epsilon}. \quad (23)$$

Proof. Note first that we need only to consider the prime factors of a less than or equal to M , since for $1 \leq n \leq M$, $(n, a) = 1 \Leftrightarrow (n, a_M) = 1$.

To calculate our sum we will write it as a contour integral and shift contours, showing that the contribution of the shifted contours is negligible and obtaining the main terms from the residues at the poles.

Setting $\kappa = \frac{1}{\log M}$ in Lemma 5.8,

$$\begin{aligned} \sum_{\substack{n \leq M \\ (n, a) = 1}} \frac{1}{\phi(n)} \left(1 - \frac{n}{M}\right) &= \sum_{(n, a_M) = 1} \frac{1}{\phi(n)} h\left(\frac{M}{n}\right) \\ &= \sum_{(n, a_M) = 1} \frac{1}{\phi(n)} \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \left(\frac{M}{n}\right)^s \frac{ds}{s(s+1)} \\ &\quad + O\left(\frac{1}{T^2} \sum_{n \neq M} \frac{1}{\phi(n) |\log M/n|} \left(\frac{M}{n}\right)^{\kappa} + \frac{\log M}{TM}\right) \\ &= \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \left(\sum_{(n, a_M) = 1} \frac{1}{n^s \phi(n)}\right) \frac{M^s}{s(s+1)} ds + O_M\left(\frac{1}{T}\right). \end{aligned} \quad (24)$$

In the last step we used the elementary estimates

$$\sum_{n \leq M} \frac{1}{\phi(n)} \ll \log M \quad \text{and} \quad \sum_{n > M} \frac{1}{n^{\kappa} \phi(n)} \ll \log M.$$

Now taking Euler products we compute that

$$\sum_{(n, a_M) = 1} \frac{1}{n^s \phi(n)} = \mathfrak{S}_{a_M}(s) \zeta(s+1) \zeta(s+2) Z(s) \quad (25)$$

where

$$\mathfrak{S}_{a_M}(s) := \prod_{\substack{p|a \\ p \leq M}} \left(1 - \frac{1}{p^{s+1}}\right) \left(1 + \frac{1}{(p-1)p^{s+1}}\right)^{-1} \quad (26)$$

and

$$Z(s) := \prod_p \left(1 + \frac{1}{p(p-1)} \left(\frac{1}{p^{s+1}} - \frac{1}{p^{2s+2}}\right)\right), \quad (27)$$

which converges for $\Re s > -3/2$. Therefore, (24) becomes

$$\sum_{\substack{n \leq M \\ (n, a) = 1}} \frac{1}{\phi(n)} \left(1 - \frac{n}{M}\right) = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} \mathfrak{S}_{a_M}(s) \zeta(s+1) \zeta(s+2) Z(s) \frac{M^s}{s(s+1)} ds + O_M\left(\frac{1}{T}\right). \quad (28)$$

The different results for different values of $\omega(a_M)$ come from the pole at $s = -1$. We see that $\mathfrak{S}_{a_M}(s)$ has a zero of order $\omega(a_M)$ at $s = -1$ whereas

$$\frac{\zeta(s+1)\zeta(s+2)}{s(s+1)}$$

has a pole of order two at $s = -1$. Hence the product has no pole if $\omega(a_M) \geq 2$, a pole of order one if $\omega(a_M) = 1$, and a pole of order two if $\omega(a_M) = 0$. We now shift the contour of integration to the left until the line $\Re(s) = \sigma$, where $-1 - \frac{1}{2+4\theta} < \sigma < -1$ and $\theta := \frac{32}{205}$. The right hand side of (28) becomes

$$= P_T + \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} \mathfrak{S}_{a_M}(s) \zeta(s+1) \zeta(s+2) Z(s) \frac{M^s}{s(s+1)} ds + O_M\left(\frac{1}{T}\right) + O_a\left(\frac{(\log T)^2}{T^2} \left(T^{1/6} + \frac{T^{1/2}}{M^{1/2}} + \frac{T^{7/6}}{M}\right)\right). \quad (29)$$

Here, P_T denotes the sum of all residues in the box $\sigma \leq \Re s \leq \frac{1}{\log M}$ and $|\Im s| \leq T$. The second error term in (29) comes from the horizontal integrals which we have bounded using Lemma 5.6 (note that $\theta < 1/6$). Taking $T \rightarrow \infty$ yields

$$\sum_{\substack{n \leq M \\ (n, a) = 1}} \frac{1}{\phi(n)} \left(1 - \frac{n}{M}\right) = P_\infty + E(M, a), \quad (30)$$

where

$$E(M, a) := \frac{1}{2\pi i} \int_{(\sigma)} \mathfrak{S}_{a_M}(s) \zeta(s+1) \zeta(s+2) Z(s) \frac{M^s}{s(s+1)} ds.$$

Now on the line $\Re s = \sigma$ we have the bound (note that $0 < -1 - \sigma < \frac{1}{2+4\theta}$)

$$\mathfrak{S}_{a_M}(s) \ll a_M^{-1-\sigma} \prod_{p|a_M} \left(1 + \frac{1}{p^\delta}\right),$$

for some $\delta > 0$. Combining this with Lemma 5.6 yields

$$\begin{aligned}
E(M, a) &\ll_{\sigma} \prod_{p|a_M} \left(1 + \frac{1}{p^{\delta}}\right) a_M^{-1-\sigma} \int_{-\infty}^{\infty} |\zeta(\sigma + 1 + it)| |\zeta(\sigma + 2 + it)| \frac{M^{\sigma}}{(|t| + 1)^2} dt \\
&\ll \frac{\prod_{p|a_M} \left(1 + \frac{1}{p^{\delta}}\right)}{M} \left(\frac{a_M}{M}\right)^{-1-\sigma} \int_{-\infty}^{\infty} \frac{(|t| + 1)^{1/2-(\sigma+1)} (|t| + 1)^{2\theta(1-(\sigma+2))}}{(|t| + 1)^2} dt \quad (31) \\
&\ll_{\sigma} \frac{\prod_{p|a_M} \left(1 + \frac{1}{p^{\delta}}\right)}{M} \left(\frac{a_M}{M}\right)^{-1-\sigma},
\end{aligned}$$

since $1/2 - (\sigma + 1) + 2\theta(1 - (\sigma + 2)) < 1$ by our choice of σ . The claimed bound on $E(M, a)$ then follows by taking $\sigma := -1 - \frac{1}{2+4\theta} + \epsilon$ in (31).

It remains to compute P_{∞} which is the sum of the residues of $\mathfrak{S}_{a_M}(s)\zeta(s+1)\zeta(s+2)Z(s)\frac{M^s}{s(s+1)}$ in the region $\sigma \leq \Re s \leq \frac{1}{\log M}$. Note that $\mathfrak{S}_{a_M}(s)$ has poles on the lines $\Re s = -1 - \frac{\log(p-1)}{\log p}$, however these poles are cancelled by the zeros of $Z(s)$. Thus the only possible singularities of $\mathfrak{S}_{a_M}(s)\zeta(s+1)\zeta(s+2)Z(s)\frac{M^s}{s(s+1)}$ in the region in question are at the points $s = 0$ and $s = -1$.

Now a lengthy but straightfoward computation shows that we have a double pole at $s = 0$ with residue equal to $C_1(a_M) \log M + C_3(a_M)$.

As for $s = -1$, we have to consider three cases.

If $\omega(a_M) \geq 2$, then $\mathfrak{S}_{a_M}(s) = O((s+1)^2)$ around $s = -1$, so $\mathfrak{S}_{a_M}(s)\zeta(s+1)\zeta(s+2)Z(s)\frac{M^s}{s(s+1)}$ is holomorphic and we don't have any residue.

If $\omega(a_M) = 1$, then $\mathfrak{S}_{a_M}(s)$ has a simple zero at $s = -1$ and thus $\mathfrak{S}_{a_M}(s)\zeta(s+1)\zeta(s+2)Z(s)\frac{M^s}{s(s+1)}$ has a simple pole with residue equal to $\frac{\phi(a_M)}{a_M} \frac{\Lambda(a_M)}{2M}$.

Finally, if $a_M = 1$, then $\mathfrak{S}_{a_M}(s) \equiv 1$ and thus $\mathfrak{S}_{a_M}(s)\zeta(s+1)\zeta(s+2)Z(s)\frac{M^s}{s(s+1)}$ has a double pole at $s = -1$ with residue equal to $\frac{1}{2} \frac{\log M}{M} + \frac{C_5}{M}$. □

6. FURTHER RESULTS AND PROOFS

We will start by giving the fundamental result of this paper which works for M fixed as well as for M varying with x under the condition $M \leq (\log x)^{O(1)}$.

Proposition 6.1. *Fix $A > B > 0$ and $\lambda < \frac{1}{4}$, three positive real numbers. Let $M = M(x)$ be an integer such that $1 \leq M(x) \leq (\log x)^B$. For a in the range $0 < |a| \leq x^{\lambda}$ we have that*

$$\begin{aligned}
\sum_{\substack{q \leq \frac{x}{M} \\ (q, a) = 1}} \left(\psi(x; q, a) - \Lambda(a) - \frac{\psi(x)}{\phi(q)} \right) &= x \left(C_1(a) \log M + C_3(a) - \sum_{\substack{r \leq M \\ (r, a) = 1}} \frac{1}{\phi(r)} \left(1 - \frac{r}{M} \right) \right) \\
&\quad + O_{A, B, \lambda} \left(\frac{\phi(a)}{a} 2^{\omega(a)} \frac{x}{(\log x)^A} \right). \quad (32)
\end{aligned}$$

We can remove the condition of M being an integer at the cost of adding the error term $O(x \log \log M / M^2)$.

Proof. We will prove that

$$\sum_{\substack{q \leq \frac{x}{M} \\ (q,a)=1}} \left(\psi(x; q, a) - \Lambda(a) - \frac{x}{\phi(q)} \right) = x \left(C_1(a) \log M + C_3(a) - \sum_{\substack{r \leq M \\ (r,a)=1}} \frac{1}{\phi(r)} \left(1 - \frac{r}{M} \right) \right) + O_{A,B,\lambda} \left(\frac{\phi(a)}{a} 2^{\omega(a)} \frac{x}{(\log x)^A} \right). \quad (33)$$

From this we can deduce the proposition since by Lemma 5.5 the difference between the left hand side of (32) and that of (33) is negligible.

Define $L := (\log x)^{A+3}$. Partitioning the sum into dyadic intervals and applying Theorem 2.1 gives

$$\sum_{\substack{q \leq \frac{x}{L} \\ (q,a)=1}} \left(\psi(x; q, a) - \Lambda(a) - \frac{x}{\phi(q)} \right) = O_{A,\lambda} \left(2^{\omega(a)} \frac{x}{(\log x)^{A+1}} \right). \quad (34)$$

Therefore, we need to compute

$$\sum_{\substack{\frac{x}{L} < q \leq \frac{x}{M} \\ (q,a)=1}} \left(\psi(x; q, a) - \Lambda(a) - \frac{x}{\phi(q)} \right),$$

which by lemmas 5.3 and 5.4 is equal to

$$\sum_{\substack{\frac{x}{L} < q \leq \frac{x}{M} \\ (q,a)=1}} \left(\theta^*(x; q, a) - \frac{x}{\phi(q)} \right) + O(x^{2/3} + |a|(\log |a|)^2). \quad (35)$$

We split the sum in (35) in three distinct sums as following:

$$\sum_{\substack{\frac{x}{L} < q \leq x \\ (q,a)=1}} \theta^*(x; q, a) - \sum_{\substack{\frac{x}{M} < q \leq x \\ (q,a)=1}} \theta^*(x; q, a) - x \sum_{\substack{\frac{x}{L} < q \leq \frac{x}{M} \\ (q,a)=1}} \frac{1}{\phi(q)} = I - II - III.$$

The third sum is easily treated using Lemma 5.2:

$$\begin{aligned} III &= x \sum_{\substack{\frac{x}{L} < q \leq \frac{x}{M} \\ (q,a)=1}} \frac{1}{\phi(q)} = x \left(C_1(a) \log(x/M) + C_1(a) + C_3(a) + O \left(2^{\omega(a)} \frac{\log(x/M)}{x/M} \right) \right. \\ &\quad \left. - \left(C_1(a) \log(x/L) + C_1(a) + C_3(a) + O \left(2^{\omega(a)} \frac{\log(x/L)}{x/L} \right) \right) \right) \\ &= C_1(a)x \log(L/M) + O \left(2^{\omega(a)} L \log x \right). \quad (36) \end{aligned}$$

For the first sum, we have

$$I = \sum_{\substack{\frac{x}{L} < q \leq x \\ (q,a)=1}} \theta^*(x; q, a) = \sum_{\substack{\frac{x}{L} < q \leq x \\ (q,a)=1}} \sum_{\substack{|a| < p \leq x \\ p \equiv a \pmod{q}}} \log p.$$

Using Lemma 5.1,

$$\begin{aligned}
I &= \sum_{\substack{1 \leq r < (x-a)\frac{L}{x} \\ (r,a)=1}} \sum_{\substack{r\frac{x}{L}+a < p \leq x \\ p \equiv a \pmod r \\ p > |a|}} \log p + O(|a| \log x) \\
&= \sum_{\substack{1 \leq r < (x-a)\frac{L}{x} \\ (r,a)=1}} \left(\theta^*(x; r, a) - \theta^*\left(r\frac{x}{L} + a; r, a\right) \right) + O(x^{1/3}) \\
&= \sum_{\substack{1 \leq r < L-a\frac{L}{x} \\ (r,a)=1}} \left(\frac{x}{\phi(r)} - \frac{rx}{\phi(r)L} \right) + O(|a|L(\log x)^2) + O_A \left(L \frac{x}{L(\log x)^{A+1}} \right).
\end{aligned}$$

In the last step we used Lemma 5.4 (with $|a| < x^\lambda$) combined with the Siegel-Walfisz theorem in the form $\theta(x; r, a) - \frac{x}{\phi(r)} \ll_A \frac{x}{L(\log x)^{A+1}}$ for $r \leq 2L$, as well as the estimate $\sum_{r \leq R} \frac{|a|}{\phi(r)} \ll |a| \log R$. As $|a\frac{L}{x}| \leq 1$ for x large enough and $\phi(r) \gg r / \log \log r$, this gives

$$\begin{aligned}
I &= x \sum_{\substack{r < L \\ (r,a)=1}} \frac{1}{\phi(r)} \left(1 - \frac{r}{L} \right) + O_A \left(\frac{x}{(\log x)^{A+1}} \right) + O \left(\frac{x \log \log L}{L} \right) \\
&= x \sum_{\substack{r < L \\ (r,a)=1}} \frac{1}{\phi(r)} \left(1 - \frac{r}{L} \right) + O_A \left(\frac{x}{(\log x)^{A+1}} \right).
\end{aligned}$$

We conclude the evaluation of I by applying Lemma 5.2:

$$I = x \left(C_1(a) \log L + C_3(a) + O_A \left(\frac{2^{\omega(a)}}{(\log x)^{A+1}} \right) \right). \quad (37)$$

Now with a similar computation using lemmas 5.1 and 5.4 as well as the Siegel-Walfisz theorem in the form $\theta(x; r, a) - \frac{x}{\phi(r)} \ll_{A,B} \frac{x}{M(\log x)^{A+1}}$ for $r \leq 2M$, we show that

$$\begin{aligned}
II &= \sum_{\substack{1 \leq r < (x-a)\frac{M}{x} \\ (r,a)=1}} \left(\theta^*(x; r, a) - \theta^*\left(r\frac{x}{M} + a; r, a\right) \right) + O(|a| \log x) \\
&= \sum_{\substack{1 \leq r < M-a\frac{M}{x} \\ (r,a)=1}} \left(\frac{x}{\phi(r)} - \frac{rx}{\phi(r)M} \right) + O(|a|M(\log x)^2) + O_{A,B} \left(M \frac{x}{M(\log x)^{A+1}} \right) \\
&= x \sum_{\substack{1 \leq r \leq M \\ (r,a)=1}} \frac{1}{\phi(r)} \left(1 - \frac{r}{M} \right) + O_{A,B} \left(\frac{x}{(\log x)^{A+1}} \right).
\end{aligned}$$

In the last step we have used that M is an integer so the term for $r = M$ in the sum is given by $\frac{1}{\phi(M)} \left(1 - \frac{M}{M} \right) = 0$. If $M \notin \mathbb{N}$, we have to add the error term $\frac{x}{\phi(\lfloor M \rfloor)} \left(1 - \frac{\lfloor M \rfloor}{M} \right) = O(x \log \log M / M^2)$.

We conclude the proof by combining our estimates for I , II and III with the bound $\frac{a}{\phi(a)} \ll \log \log x$. \square

Proof of Proposition 3.2. Follows from Proposition 6.1. \square

For M not necessarily fixed, we will need to use Lemma 5.9.

Theorem 6.2. *Let $M = M(x) \leq (\log x)^B$ (not necessarily an integer) where $B > 0$ is a fixed real number. Fix $\lambda < \frac{1}{4}$ a positive real number, and let $a \neq 0$ be an integer such that $|a| \leq x^\lambda$.*

If $\omega(a_M) \geq 1$,

$$\frac{1}{\frac{\phi(a)}{a} \frac{x}{M}} \sum_{\substack{q \leq \frac{x}{M} \\ (q,a)=1}} \left(\psi(x; q, a) - \Lambda(a) - \frac{\psi(x)}{\phi(q)} \right) = -\frac{1}{2} \Lambda(a_R) + E(M, a). \quad (38)$$

If $a_M = 1$,

$$\frac{1}{\frac{\phi(a)}{a} \frac{x}{M}} \sum_{\substack{q \leq \frac{x}{M} \\ (q,a)=1}} \left(\psi(x; q, a) - \Lambda(a) - \frac{\psi(x)}{\phi(q)} \right) = -\frac{1}{2} \log M - C_5 + E(M, a). \quad (39)$$

Under the assumption that $\sum_{\substack{p|a \\ p > M}} \frac{1}{p} \leq 1$, the error term $E(M, a)$ satisfies, for some $\delta > 0$,

$$E(M, a) \ll_{a, \epsilon, B} \frac{a}{\phi(a)} \prod_{p|a_M} \left(1 + \frac{1}{p^\delta} \right) \left(\frac{a_M}{M} \right)^{\frac{205}{538} - \epsilon} + M \log M \sum_{\substack{p|a \\ p > M}} \frac{\log p}{p} + \frac{2^{\omega(a)}}{(\log x)^{B+1}} + \frac{a}{\phi(a)} \frac{\log \log M}{M}. \quad (40)$$

Remark 6.3. If we assume Lindelöf's hypothesis, we can replace the first term of the right hand side of (40) by $\frac{a}{\phi(a)} \prod_{p|a_M} \left(1 + \frac{1}{p^\delta} \right) \left(\frac{a_M}{M} \right)^{1/2 - \epsilon}$.

Proof of Theorem 6.2. Our starting point will be to set $A := 2B + 1$ in Proposition 6.1:

$$\sum_{\substack{q \leq \frac{x}{M} \\ (q,a)=1}} \left(\psi(x; q, a) - \Lambda(a) - \frac{\psi(x)}{\phi(q)} \right) = x \left(C_1(a) \log M + C_3(a) - \sum_{\substack{r \leq M \\ (r,a)=1}} \frac{1}{\phi(r)} \left(1 - \frac{r}{M} \right) \right) + O_B \left(\frac{\phi(a)}{a} \frac{x}{M} \frac{2^{\omega(a)}}{(\log x)^{B+1}} + x \frac{\log \log M}{M^2} \right). \quad (41)$$

We now consider three cases depending on the number of prime factors of a_M .

Case 1: $\omega(a_M) \geq 2$, which implies $\omega(a) \geq 2$. Applying Lemma 5.9 gives

$$\sum_{\substack{q \leq \frac{x}{M} \\ (q,a)=1}} \left(\psi(x; q, a) - \Lambda(a) - \frac{\psi(x)}{\phi(q)} \right) \ll_B x \left(|C_1(a_M) - C_1(a)| \log M + |C_3(a_M) - C_3(a)| \right. \\ \left. + \frac{\prod_{p|a_M} \left(1 + \frac{1}{p^\delta} \right)}{M} \left(\frac{a_M}{M} \right)^{\frac{205}{538} - \epsilon} \right) + \frac{\phi(a)}{a} \frac{x}{M} \frac{2^{\omega(a)}}{(\log x)^{B+1}} + x \frac{\log \log M}{M^2}. \quad (42)$$

If all the prime factors of a are less than or equal to M , then $|C_i(a_M) - C_i(a)| = 0$. If not, we need upper bounds. By the definition of $C_1(a)$,

$$\begin{aligned} |C_1(a_M) - C_1(a)| &= C_1(a) \left(\prod_{\substack{p|a \\ p > M}} \left(1 - \frac{1}{p}\right)^{-1} \left(1 - \frac{1}{p^2 - p + 1}\right)^{-1} - 1 \right) \\ &\ll \frac{\phi(a)}{a} \sum_{\substack{p|a \\ p > M}} \frac{1}{p}, \end{aligned}$$

as long as $\sum_{\substack{p|a \\ p > M}} \frac{1}{p} \leq 1$. Moreover,

$$\begin{aligned} C_3(a_M) - C_3(a) &= \left(C_1(a) + O\left(\frac{\phi(a)}{a} \sum_{\substack{p|a \\ p > M}} \frac{1}{p}\right) \right) \frac{C_3(a_M)}{C_1(a_M)} - C_3(a) \\ &= C_1(a) \left(\frac{C_3(a_M)}{C_1(a_M)} - \frac{C_3(a)}{C_1(a)} \right) + O\left(\frac{\phi(a)}{a} \sum_{\substack{p|a \\ p > M}} \frac{\log M}{p}\right), \end{aligned}$$

since $\sum_{p|a_M} \frac{p^2 \log p}{(p-1)(p^2-p+1)} \ll \sum_{p \leq M} \frac{\log p}{p} \ll \log M$. Thus,

$$\begin{aligned} |C_3(a_M) - C_3(a)| &\ll \frac{\phi(a)}{a} \sum_{\substack{p|a \\ p > M}} \frac{p^2 \log p}{(p-1)(p^2-p+1)} + \frac{\phi(a)}{a} \sum_{\substack{p|a \\ p > M}} \frac{\log M}{p} \\ &\ll \frac{\phi(a)}{a} \sum_{\substack{p|a \\ p > M}} \frac{\log p}{p}. \end{aligned}$$

Putting all this together and dividing by $\frac{x}{M} \frac{\phi(a)}{a}$ gives the claimed estimate.

Case 2: $\omega(a_M) = 1$. The calculation is similar, however Lemma 5.9 gives a contribution of

$$-\frac{1}{2} \frac{a}{\phi(a)} \frac{\phi(a_M)}{a_M} \Lambda(a_M) = -\frac{1}{2} \Lambda(a_M) + O\left(\log M \sum_{\substack{p|a \\ p > M}} \frac{1}{p}\right),$$

since $\Lambda(a_M) \leq \log M$.

Case 3: $a_M = 1$. The contribution of

$$\frac{a}{\phi(a)} \left(-\frac{1}{2} \log M - C_5 \right) = -\frac{1}{2} \log M - C_5 + O\left(\log M \sum_{\substack{p|a \\ p > M}} \frac{1}{p}\right)$$

comes from Lemma 5.9. □

Remark 6.4. One can give an complex analytic explanation of why the main terms all cancel out in the proofs of Proposition 6.1 and Theorem 6.2. Indeed, it amounts to the fact

that the Mellin transform

$$\psi(s) := \frac{L^s - M^s}{s+1} + \left(\frac{x}{L}\right)^s - \left(\frac{x}{M}\right)^s$$

has a double zero at $s = 0$ (see the proof of Proposition 5.1 of [2]).

Proof of Theorem 1.1. It is a particular case of Theorem 6.2. The theorem is trivial if M is bounded. For M tending to infinity with x , we have that $a_M = a$ for x large enough, since a is fixed. Hence, the error term in Theorem 6.2 satisfies $E(M, a) \ll_{a, \epsilon, B} M^{-\frac{205}{538} + \epsilon}$. \square

7. CONCLUDING REMARKS

Remark 7.1. One could ask if the results of Theorem 1.1 are intrinsic to the sequence of prime numbers or if they are just a result of the weight $\Lambda(n)$ in the prime counting functions. However one can see that if we replace $\psi(x; q, a)$ by $\pi(x; q, a)$ and $\psi(x)$ by $\pi(x)$, the proof of Proposition 6.1 will go through with x replaced by $Li(x)$ and an additional error term of

$$O\left(x \frac{\log \log x}{(\log x)^2}\right).$$

(This is because in the evaluation of II in the proof of Proposition 6.1, we have for $\frac{1}{\log x} \leq y \leq 1 + o(1)$ that $Li(yx) - yLi(x) \ll \frac{|\log y|}{(\log x)^2}$, a bound which cannot be improved for small values of y .) One has to prove the analogue of Theorem 2.1 which can be done using the Bombieri-Vinogradov theorem and a very delicate summation by parts. We conclude that an analogue of Theorem 1.1 holds with the natural prime counting functions in the range $M = o\left(\frac{\log x}{\log \log x}\right)$.

Theorem 7.2. Fix an integer $a \neq 0$ and $\epsilon > 0$. If $M = M(x)$ is such that $M = o\left(\frac{\log x}{\log \log x}\right)$, then

$$\frac{1}{\frac{\phi(a)}{a} \frac{Li(x)}{M}} \sum_{\substack{q \leq \frac{x}{M} \\ (q, a) = 1}} \left(\pi(x; q, a) - \frac{\vartheta(a)}{\log a} - \frac{\pi(x)}{\phi(q)} \right) = \mu(a, M) + O_{a, \epsilon} \left(\frac{1}{M^{\frac{205}{538} - \epsilon}} + \frac{M \log \log x}{\log x} \right),$$

where $\mu(a, M)$ is defined as in Theorem 1.1.

Remark 7.3. Using Theorem 6.2, one can show that Theorem 1.1 holds for many (not necessarily fixed) values of a :

Theorem 7.4. Fix B , $\lambda < \frac{1}{4}$ and $\eta \leq \frac{1}{5}$, three positive real numbers. We have for $M \leq (\log x)^B$ that the proportion of integers a in the range $0 < |a| \leq x^\lambda$ for which

$$\frac{1}{\frac{\phi(a)}{a} \frac{x}{M}} \sum_{\substack{q \leq \frac{x}{M} \\ (q, a) = 1}} \left(\psi(x; q, a) - \frac{\psi(x)}{\phi(q)} \right) = O_{B, \lambda} \left(\frac{1}{M^\eta} \right) \quad (43)$$

is at least $\frac{1 - \frac{21}{8}\eta}{1 + \eta + \eta^2} e^{-\gamma}$, where γ is the Euler-Mascheroni constant.

We now sketch an argument showing that the proportion of integers $a \leq x^\lambda$ for which the first term of the right hand side of (40) is ≤ 1 is not more than $e^{-\gamma}$. Setting $X := x^\lambda$ and using that $M \ll (\log X)^{O(1)}$, we compute

$$\#\left\{a \leq X : \prod_{\substack{p|a \\ p \leq M}} p \leq M\right\} = \sum_{\substack{n \leq M \\ \mu^2(n)=1}} \#\left\{a \leq X : n \mid a, \left(\frac{a}{n}, \prod_{\substack{p \leq M \\ p \nmid n}} p\right) = 1\right\},$$

which by the fundamental lemma of the combinatorial sieve is

$$\sim \sum_{\substack{n \leq M \\ \mu^2(n)=1}} \frac{X}{n} \prod_{\substack{p \leq M \\ p \nmid n}} \left(1 - \frac{1}{p}\right) \sim \frac{Xe^{-\gamma}}{\log M} \sum_{n \leq M} \frac{\mu^2(n)}{\phi(n)} \sim Xe^{-\gamma}.$$

Therefore, to extend the proportion of integers a for which we get an admissible error term in Theorem 7.4, one would need to improve the bound (40) for $E(M, a)$.

It would be interesting to either extend this proportion to 1, or to show that this is impossible, that is to show that the left hand side of (43) is $\Omega(1)$ for a positive proportion of a in the range $0 < |a| \leq x^\lambda$.

Remark 7.5. If we consider dyadic sums, that is we sum over the moduli $\frac{x}{2M} < q \leq \frac{x}{M}$ with $(q, a) = 1$ as in (44), then we can improve the range of a in our results to $|a| \leq \frac{x}{(\log x)^C}$ for a large enough constant C , since in this case we do not need to appeal to Theorem 2.1. (It was already remarked by the authors in [4] that if Q is not in the interval $[x^{\frac{1}{2}-\epsilon}, x^{\frac{1}{2}+\epsilon}]$, then Theorem 2.1 holds in a much wider range of a .)

Remark 7.6. If we assume GRH and we use dyadic intervals, then we can improve the range of M in Theorem 1.1. Indeed we can show that for $M = M(x) = o(x^{\frac{1}{4}}/\log x)$ and for any $\epsilon > 0$,

$$\begin{aligned} \frac{1}{\frac{\phi(a)}{a} \frac{x}{2M}} \sum_{\substack{\frac{x}{2M} < q \leq \frac{x}{M} \\ (q, a)=1}} \left(\psi(x; q, a) - \Lambda(a) - \frac{\psi(x)}{\phi(q)} \right) &= 2\mu(a, M) - \mu(a, 2M) \\ &+ O_{a, \epsilon} \left(\frac{M^2}{\sqrt{x}} (\log x)^2 + \frac{1}{M^{\frac{1}{2}-\epsilon}} \right). \end{aligned} \quad (44)$$

If we assume Montgomery's hypothesis, then we can show that the error term in (44) is $o(1)$ for $M \leq x^{\frac{1}{3}-\epsilon}$, for any fixed $\epsilon > 0$.

Remark 7.7. What the proof of Theorem 1.1 boils down to is the distribution of primes in arithmetic progressions, rather than depending on the primes themselves. For this reason, one can generalize Theorem 1.1 to many arithmetic sequences that are well distributed in arithmetic progressions. This will be done in a forthcoming article [2].

REFERENCES

- [1] Enrico Bombieri, John B. Friedlander and Henryk Iwaniec, *Primes in arithmetic progressions to large moduli*. Acta Math. 156 (1986), no. 3-4, 203–251.
- [2] Daniel Fiorilli, *The influence of the first term of an arithmetic progression*, preprint. <http://arxiv.org/abs/1104.2542>.
- [3] Étienne Fouvry, *Sur le problème des diviseurs de Titchmarsh*. J. Reine Angew. Math. 357 (1985), 51–76.

- [4] John B. Friedlander, Andrew Granville, *Relevance of the residue class to the abundance of primes*. Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), 95–103, Univ. Salerno, Salerno, 1992.
- [5] John B. Friedlander, Andrew Granville, Adolf Hildebrand, Helmut Maier, *Oscillation theorems for primes in arithmetic progressions and for sifting functions*. J. Amer. Math. Soc. 4 (1991), no. 1, 25–86.
- [6] Christopher Hooley, *On the Barban-Davenport-Halberstam theorem. I*. Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, III. J. Reine Angew. Math. 274/275 (1975), 206–223
- [7] Martin N. Huxley, *Exponential sums and the Riemann zeta function. V*. Proc. London Math. Soc. (3) 90 (2005), no. 1, 1–41.
- [8] Gérald Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres*. Deuxième édition, Cours Spécialisés, 1. Société Mathématique de France, Paris, 1995. xv+457 pp.

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